Probability Review

CSC 343·643

WAKE FOREST UNIVERSITY

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Basic Definitions

- **Random experiment** - Observing the outcome of a chance event
- **Elementary outcomes** - All possible results of the experiment
- **Sample space** - Set of all the elementary outcomes
- Consider tossing a coin
  - Random experiment consists of recording the outcome
  - Elementary outcomes are heads and tails
  - Sample space is $\{\text{heads}, \text{tails}\}$

  What about rolling a single die?

Probability

- We want to assign a numerical weight to each outcome
  - This **probability** measures the likelihood of it occurring
  - Denoted as $p\{x_i\}$, which is the probability of x_i
- Consider rolling a fair die and recording the side facing up
  - Since the die is fair, all sides have an equal probability
  - $p\{1\} = p\{2\} = p\{3\} = p\{4\} = p\{5\} = p\{6\} = \frac{1}{6}$
- Of course a *loaded* die would have one side appear more often
  - $p\{1\} = \frac{1}{4}$
  - $p\{2\} = p\{3\} = p\{4\} = p\{5\} = p\{6\} = \frac{3}{20}$

    What is the assumption for the remaining sides?

Characteristic Properties of Probability

- Probabilities are never negative
  - $p\{x_i\} \geq 0$
  - Probability of zero means the event cannot happen
- Probabilities are never greater than 1
  - $p\{x_i\} \leq 1$
  - Probability of 1 means the event is certain to happen
- The total probability of all possible outcomes is one

$$\sum_{i=1}^{n} p\{x_i\} = p\{x_1\} + ... + p\{x_n\} = 1$$

Did the previous examples adhere to these rules?

Events

- An **event** is a set of elementary outcomes
  - The probability of an event is the sum of the probabilities of the elementary outcomes
- Consider rolling a pair of dice

Event	Elementary Outcomes	Probability
A = dice add to three	$\{(1,2),(2,1)\}$	$p\{A\} = \frac{2}{36}$
B = dice add to six	$\{(1,5),(2,4),(3,3),(4,2),(5,1)\}$	$p\{B\} = \frac{5}{36}$
C = first die shows 1	$\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\}$	$p\{C\} = \frac{6}{36}$
D = second die shows 1	$\{(1,1),(2,1),(3,1),(4,1),(5,1),(6,1)\}$	$p\{D\} = \frac{6}{36}$

Where did the denominator of 36 come from?

Combining Events

- Given two events, we can combine them to create a new event
  - E **and** F, the event E and the event F both occur
  - E **or** F, the event E or the event F (or both) occur
  - **not** E, the event E does not occur
- Remember events can be represented as sets
  - Combining events is really just set manipulation

Addition Rule

- For any events E and F,

$$p\{\text{E or F}\} = p\{\text{E}\} + p\{\text{F}\} - p\{\text{E and F}\}$$

  – Adding *double counts* the elementary outcomes shared by E and F, so we must subtract the extra amount

- *What is the probability of the first or second die being one*?

$$p\{\text{C or D}\} = p\{\text{C}\} + p\{\text{D}\} - p\{\text{C and D}\} = \frac{6}{36} + \frac{6}{36} - \frac{1}{36} = \frac{11}{36}$$

- If E and F are mutually exclusive, then the two events will not *double count*, as a result

$$p\{\text{E or F}\} = p\{\text{E}\} + p\{\text{F}\}$$

What is a dice example of this?

Multiplication Rule

- For any events E and F,

$$p\{\text{E and F}\} = p\{\text{E}|\text{F}\} \cdot p\{\text{F}\}$$

  – Where $p\{\text{E}|\text{F}\}$ is the conditional probability of E will occur given that F has already occurred

- If E and F are independent, then the occurrence of one has no influence on the other

$$p\{\text{E and F}\} = p\{\text{E}\} \cdot p\{\text{F}\}$$

- *What is the probability of snake eyes?*

$$p\{\text{both die are 1}\} = p\{\text{first die is 1}\} \cdot p\{\text{second die is 1}\} =$$

$$\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

Subtraction Rule

- For any event E,

$$p\{\text{not E}\} = 1 - p\{\text{E}\}$$

- *What is the probability of not rolling snake eyes?*

$$p\{\text{both die are not 1}\} = 1 - p\{\text{both die are 1}\} =$$

$$1 - \frac{1}{36} = \frac{35}{36}$$

Random Variables

- A random variable, X, is the numerical outcome of an experiment
- For example X could represent
  - The sum of the faces of a pair of dice
  - Number of heads when two coins are tossed
  - Number of bits transmitted in error in a frame
- We want to observe the probabilities of the outcomes
  - The probability of the RV X having the value x

$$Pr\{X = x\} = p\{x\}$$

RV Examples

- Consider tossing two coins, let $X =$ number of heads

x	0	1	2
$Pr\{X = x\}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

How did we get these probability values?

- Consider rolling a pair of dice, let $X =$ sum of the dice

x	2	3	4	5	6	7	8	9	10	11	12
$Pr\{X = x\}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Mean of a RV

- The mean of a random variable X is

$$\sum_{\forall x} x_i \cdot p\{x_i\}$$

  - Sum of the possible values, each weighted by its probability
- Is also called the **expected value** of X, $e[X]$
- The expected number of heads when tossing two coins is

$$e[X] = \sum_{i=0}^{2} x_i \cdot p\{x_i\} = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

Bernoulli Trial

- An experiment (trial) that has the following properties
  1. Result of a trial is either a *success* or a *failure*
  2. The probability p of success is the same in every trial
  3. Independence, one trial has no influence on later outcomes
- Often used in network modeling

  What are some other example experiments that can be considered a Bernoulli trial?

Binomial RV

- X is the number of successes in n repeated Bernoulli trials with probability p
- For example, number of heads in two flips of a coin, $(n = 2, p = \frac{1}{2})$

$k =$ number of successes	0	1	2
$Pr\{X = k\}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

How did we get the probability values?

- The equation is

$$Pr\{X = k\} = \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k}$$

- Where

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$$

  – Is called n choose k, which counts the possible ways of getting k successes in n trials

Example problem

Three bits are grouped together to form a frame and transmitted over a channel. Assuming the probability of a single bit error is 0.25 and errors are independent...

- The elementary outcomes are
  – Let b represent the orginal bit, e represent an error

Bit Pattern	Probability
[bbb]	$\frac{3}{4}\frac{3}{4}\frac{3}{4} = 0.421875$
[ebb]	$\frac{1}{4}\frac{3}{4}\frac{3}{4} = 0.140625$
[beb]	$\frac{3}{4}\frac{1}{4}\frac{3}{4} = 0.140625$
[bbe]	$\frac{3}{4}\frac{3}{4}\frac{1}{4} = 0.140625$
[eeb]	$\frac{1}{4}\frac{1}{4}\frac{3}{4} = 0.046875$
[bee]	$\frac{3}{4}\frac{1}{4}\frac{1}{4} = 0.046875$
[ebe]	$\frac{1}{4}\frac{3}{4}\frac{1}{4} = 0.046875$
[eee]	$\frac{1}{4}\frac{1}{4}\frac{1}{4} = 0.015625$

- *What is the probability that a frame has at least one error?*
  - Could use *brute force*

$$p\{\text{at least one error}\} = p\{\texttt{[ebb]}\} + p\{\texttt{[beb]}\} + p\{\texttt{[bbe]}\}+$$

$$p\{\texttt{[eeb]}\} + p\{\texttt{[bee]}\} + p\{\texttt{[ebe]}\} + p\{\texttt{[eee]}\} = 0.578125$$

  - Could use Binomial RV

$$p\{\text{at least one error}\} = p\{\text{1 error}\} + p\{\text{2 errors}\} + p\{\text{3 errors}\} =$$

$$\binom{3}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^2 + \binom{3}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^1 + \binom{3}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^0$$

$$= 0.578125$$

  - Could use *not*

$$p\{\text{at least one error}\} = 1 - p\{\text{no error}\}$$

$$= 1 - 0.421875 = 0.578125$$

- *Assume the third bit is a parity bit, what is the probability of an undetected error?* Parity cannot detect an even number of errors
  - Brute force

$$p\{\text{2 errors}\} = p\{\texttt{[eeb]}\} + p\{\texttt{[bee]}\} + p\{\texttt{[ebe]}\} = 0.140625$$

  - Binomial RV

$$\binom{3}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^1 = 0.140625$$

- *What is the average number of transmissions required to receive the frame with no errors?*
  - Let $g = p\{\text{error in frame}\}$, $1 - g = p\{\text{no error in frame}\}$, and i be a RV that represents the number of transmissions
  - If i transmissions are required, then we would have $(i-1)$ failures followed by 1 success
  - Use the expected RV formula

$$e[i] = \sum_{i=0}^{\infty} i \cdot p\{\text{i transmissions}\} = \sum_{i=0}^{\infty} i \cdot g^{i-1} \cdot (1-g)$$

$$= \frac{1}{1-g} = \frac{1}{1-0.578125} = 2.37$$